

TRACKING EIGENVALUES TO THE FRONTIER OF MODULI SPACE I: CONVERGENCE AND SPECTRAL ACCUMULATION

CHRISTOPHER M. JUDGE

For Ralph Phillips

ABSTRACT. We consider the spectral behavior of the Laplace-Beltrami operator associated to a class of singular perturbations of a Riemannian metric on a complete manifold. The class of perturbations generalizes the well-known ‘opening node’ perturbation of Teichmüller theory. In particular, we recover results of Ji and Zworski [JiZwr93] and Wolpert [Wlp92] from our more general methods.

1. INTRODUCTION

Let M and N be compact differentiable manifolds of dimension d and $d+1$ respectively. Given a compact interval $I \subset \mathbb{R}$, we suppose that we have a (fixed) embedding $I \times M \subset N$. See Figure 1. Let N^0 denote the complement of $\{0\} \times M$.

Let h be a Riemannian metric on M , and let ρ be a positive function on \mathbb{R}^2 that is positively homogeneous of degree 1 and smooth away from $(0,0)$. Given $(a,b) \in \mathbb{R}^2$, we consider continuous families, g_ϵ , of symmetric $(0,2)$ -tensors on N such that

$$(1) \quad g_\epsilon|_{I \times M} = \rho(\epsilon, t)^{2a} \cdot dt^2 + \rho(\epsilon, t)^{2b} \cdot h$$

and such that g_ϵ is positive definite on N^0 . Note that for $\epsilon \neq 0$, the tensor g_ϵ is a Riemannian metric on all of N , but that the tensor g_0 is singular on $\{0\} \times M$ if $(a,b) \neq (0,0)$.

Example 1.1 (Hyperbolic Degeneration). Let γ be a simple closed curve on a compact oriented surface N with $\chi(N) < 0$. Let g_ϵ be a metric on N of constant curvature -1 such that the unique geodesic homotopic to γ has length $\epsilon < 2 \cosh^{-1}(2)$. By the collar lemma [Bsr], there exists an embedding $I \times \gamma \rightarrow N$ with $I = [-1, 1]$ such that¹

$$(2) \quad g_\epsilon|_{I \times \gamma} = \frac{dt^2}{\epsilon^2 + t^2} + (\epsilon^2 + t^2) dx^2$$

where x is the usual coordinate on the circle $\mathbb{R}/\mathbb{Z} \cong \gamma$. Note that the Riemannian surface $((I \times \gamma)^0, g_0)$ is a union of hyperbolic cusps.

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¹ The coordinates given here for a collar made their first appearance in [JdgPhl97]. To obtain the more common Fermi coordinates, let $t = \epsilon \sinh(\rho)$.

In both this paper and [Jdg00] we study the small ϵ behavior of the spectrum of the Laplace-Beltrami operator Δ_ϵ associated to a metric families g_ϵ as described above in (1). *In this paper, we restrict our attention to (a, b) satisfying $a \leq -1$ and $b > 0$.*

Theorem 1.2 (Theorem 5.1). *Let $\epsilon_j \rightarrow 0$. Any sequence ψ_j of eigenfunctions of Δ_{ϵ_j} with uniformly bounded eigenvalues has a subsequence that converges (up to rescaling) to an eigenfunction $\psi_* \neq 0$ of Δ_0 . In particular, for each compact subset of $A \subset N^0$ the subsequence converges to ψ_* in $H^1(A, dV_0)$.*

In the special case of Example 1.1, the preceding theorem was obtained by Wolpert [Wlp92] and Ji [Ji93].

As observed in [Mlr] §8.1, the manifold (N^0, g_0) is Riemannian complete if and only if $a \leq -1$. In §4, we prove that if g_0 is ‘marginally complete’, $a = -1$, then the essential spectrum of Δ_0 consists of the band $[(2^{-1}bdc)^2, \infty[$ where c is determined by ρ . See Proposition 4.2. In the ‘overcomplete’ case, $a < -1$, we show that the essential spectrum consists of the band $[0, \infty[$.

On the other hand, for $\epsilon \neq 0$, the operator Δ_ϵ has purely discrete spectrum. Hence one is led to ask about the nature of the transition from discrete to continuous spectrum as ϵ tends to zero. The following gives a precise quantitative description of this transition.

Theorem 1.3 (Theorem 7.1). *Let $a = -1$, $b > 0$, and let c_\pm satisfy $\rho(0, t) = c_\pm t$ for $\pm t > 0$. For $\Lambda > 0$, let $\mathcal{N}_\Lambda(\epsilon)$ be the number of eigenvalues of Δ_ϵ that lie in $[0, \Lambda]$. Then*

$$\mathcal{N}_\Lambda(\epsilon) = \left(c_+ \sqrt{\Lambda - \left(\frac{c_+ \cdot b \cdot d}{2} \right)^2} + c_- \sqrt{\Lambda - \left(\frac{c_- \cdot b \cdot d}{2} \right)^2} \right) \cdot \frac{\log(\epsilon^{-1})}{\pi} + O_\Lambda(1).$$

L. Ji and M. Zworski [JiZwr93] obtained Theorem 1.3 in the special case of Example 1.1 following the earlier work of [Ji93] and [Wlp87].

We now provide an outline of the present paper. In §2 we establish notation. In §3 we separate the action of the Laplacian on functions that are constant on each fibre of $I \times M \rightarrow I$ from its action on those functions whose integral along each fibre vanishes. Using a classical transformation of Sturm-Liouville theory, we demonstrate a unitary equivalence of the former action with the action of

$$\frac{\partial^2}{\partial s^2} - \frac{bd}{2} \cdot (\alpha^{-1})^* \left(\rho^{-2a-2} \left(\rho \cdot \rho''_{tt} + \frac{bd-2a-2}{2} (\rho'_t)^2 \right) \right)$$

on $L^2(\alpha(I), ds)$ where α is a diffeomorphism of intervals determined by ρ (Lemma 3.2). We also provide a convexity estimate on the fibrewise $L^2(M, dV_h)$ -norm of Laplace eigenfunctions whose integral along each fibre is zero (Lemma 3.4).

In §4 we obtain the basic spectral theory of a manifold with (generalized) cusps by specializing the results of §3 to the case $\epsilon = 0$. For example, we show that the ‘cut-off’ Laplacian Δ_0^\perp is compactly resolved, and that the fibrewise $L^2(M, dV_h)$ -norm of a cusp form vanishes to arbitrary order at $t = 0$.

In §5 we prove Theorem 1.2 using the results of earlier sections. In §6 we show that the spectrum of Δ_ϵ^\perp varies continuously in ϵ including at $\epsilon = 0$. We use

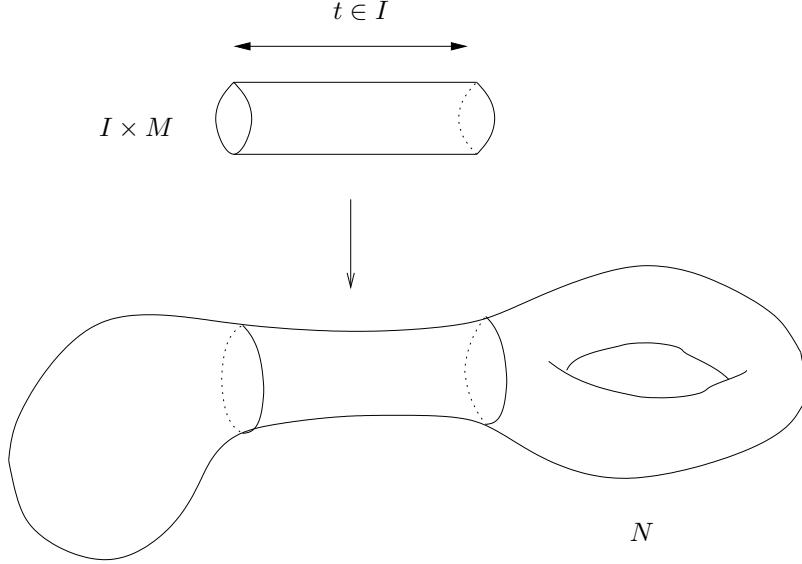


FIGURE 1. The set up

this continuity in §7 together with Dirichlet-Neumann bracketing and elementary Sturm-Liouville theory to prove Theorem 1.3.

2. PRELIMINARIES AND NOTATION

Let g be a Riemannian metric on a manifold N , and let dV_g and ∇_g denote respectively the associated density and gradient.² The Laplace-Beltrami operator $\Delta_g : L^2(N, dV_g) \rightarrow L^2(N, dV_g)$ is defined via the Friedrichs extension³ of the form $Q(\phi, \psi) = \int_N g(\nabla\phi, \nabla\psi) dV_g$ with respect to the $L^2(N, dV_g)$ -norm. In particular, by design

$$(3) \quad \int_N g(\nabla_g \phi, \nabla_g \psi) dV_g = \int_N \phi \cdot \Delta_g \psi dV_g$$

for any ϕ and ψ in the domain of Δ_g . The core used in this Friedrichs extension is the space of smooth functions f such that $Q(f, f) + \int_N f^2 < \infty$ satisfying symmetric boundary conditions.

Consider a metric g on $I \times M$ of the form given in (1), and recall that d denotes the dimension of M . For any $f \in C_0^\infty(I \times M)$

$$(4) \quad \Delta_g f = -L(f) + \rho^{-2b} \Delta_h f$$

where

$$(5) \quad L(f) = \rho^{-a-bd} \partial_t \rho^{-a+bd} \partial_t f.$$

and

$$(6) \quad \nabla_g f = \rho^{-2a} \cdot \partial_t f + \rho^{-2b} \cdot \nabla_h f.$$

²We will drop the subscript g if it is clear from the context.

³See, for example, [Kat] page 325.

The volume form restricted to $I \times M$ is

$$(7) \quad dV_g = \rho^{a+bd} dt dV_h.$$

The function $t \rightarrow \rho(0, t)$ is positive and positively homogeneous of degree 1. Hence there exist unique positive constants c_+ and c_- such that

$$(8) \quad \rho(0, t) = \begin{cases} c_+t, & t > 0 \\ c_-t, & t < 0. \end{cases}$$

We will refer to c_{\pm} as *constants of homogeneity*.

3. THE CONSTANT MODE AND ITS COMPLEMENT

Remark 3.1. In this section, if $\epsilon = 0$, then replace I with $I \setminus \{0\}$.

Viewing $I \times M$ as an M -fibre bundle over I , we use integration along the fibres to analyse Laplace eigenfunctions on $I \times M$. For $f \in L^2(I \times M)$ define the *constant mode (zeroth Fourier coefficient) of f* to be

$$(9) \quad f_0(t) = \int_{\{t\} \times M} f(t, m) dV_h.$$

One may regard f_0 as a function on $I \times M$ that is constant along the fibres $\{t\} \times M$. From this point of view $P_0(f) = f_0$ defines an orthogonal projection onto a closed subspace of $L^2(I \times M, dV_g)$. Let P^{\perp} be the orthogonal projection onto the complementary subspace $L^2_{\perp}(I \times M, dV_g)$.

These projections diagonalize the Laplacian:

$$(10) \quad \Delta_g = P_0 \circ \Delta_g \circ P_0 + P_{\perp} \circ \Delta_g \circ P_{\perp}.$$

The operator $P_0 \Delta_g P_0$ is unitarily equivalent to the operator $-L$ (densely) defined on $L^2(I, \rho^{a+bd} dt)$ via the Friedrichs extension. In particular, if $\Delta\psi = \lambda\psi$, then

$$(11) \quad -L \psi_0 = \lambda \cdot \psi_0.$$

The operator $P_{\perp} \Delta_g P_{\perp}$ is unitarily equivalent to the restriction, $\Delta_{\epsilon}^{\perp}$, of $\Delta_{g_{\epsilon}}$ to $L^2_{\perp}(I \times M, dV_g)$.

To analyse solutions to (11), we conjugate L with a unitary operator found in classical Sturm-Liouville theory (see, for example, [CrnHlb] V §3.3). For $t_0 \in I$ and ϵ fixed, let

$$(12) \quad \alpha(t) = \int_{t_0}^t \rho^a(\epsilon, u) du.$$

and define $U : C^{\infty}(\alpha(I)) \rightarrow C^{\infty}(I)$ by

$$(13) \quad U(f) = \rho^{-\frac{bd}{2}} \cdot \alpha^*(f).$$

Proposition 3.2. *The map U extends to a unitary operator from $L^2(\alpha(I), ds)$ onto $L^2(I, \rho^{n-2} dt)$. Moreover,*

$$(14) \quad U^{-1} \circ L \circ U = \frac{\partial^2}{\partial s^2} - \frac{bd}{2} \cdot (\alpha^{-1})^* \left(\rho^{-2a-2} \left(\rho \cdot \rho''_{tt} + \frac{bd-2a-2}{2} (\rho'_t)^2 \right) \right).$$

Proof. The first claim is straightforward. The second claim is a lengthy but straightforward computation. \square

The fibrewise $L^2(\{t\} \times M, dV_h)$ -norm of a function $f \in C^0(I \times M)$ is the function on I defined by

$$(15) \quad \|\|f\|\|_M^2(t) = \int_{\{t\} \times M} f^2(t, m) dV_h(m).$$

Proposition 3.3. *Let $\psi \in C^2(I \times M)$ satisfy $\Delta_g \psi = \lambda \psi$. Then*

$$(16) \quad \frac{1}{2} L(\|\psi\|_M^2) = -\lambda \cdot \|\psi\|_M^2 + \int_{\{t\} \times M} g(\nabla \psi, \nabla \psi) dV_h.$$

Proof. Straightforward computation gives

$$\frac{1}{2} L(\|\psi\|_M^2) = \int_M L(\psi^2) dV_h = \int_M \psi L(\psi) dV_h + \rho^{-2a} \int_M (\partial_t \psi)^2 dV_h.$$

By hypothesis, we have $\int \psi \Delta \psi = \lambda \int \psi^2$ and hence by (4)

$$-\int_M \psi L(\psi) dV_h + \rho^{-2b} \int_M \psi \cdot \Delta_h \psi dV_h = \lambda \int_M \psi^2 dV_h.$$

Integrating by parts over M gives

$$(17) \quad \int_M \psi \cdot \Delta_h \psi dV_h = \int_M h(\nabla_h \psi, \nabla_h \psi) dV_h.$$

Also note that from (6) we have

$$(18) \quad g(\nabla \psi, \nabla \psi) = \rho^{-2a} (\partial_t \psi)^2 + \rho^{-2b} h(\nabla_h \psi, \nabla_h \psi).$$

The claim follows. \square

Let μ_1 denote the smallest non-zero eigenvalue of Δ_h .

Lemma 3.4 (Convexity). *For any Laplace eigenfunction ψ on $I \times M$ having constant mode $\psi_0 \equiv 0$,*

$$(19) \quad L(\|\psi\|_M^2) \geq 2 \cdot (\mu_1 \cdot \rho^{-2b} - \lambda) \cdot \|\psi\|_M^2.$$

Proof. Since $\psi_0 \equiv 0$, the function $m \rightarrow \psi(t, m)$ is orthogonal to the constants. Since the 0-eigenspace consists of the constants, by the minimax principle

$$(20) \quad \int_{\{t\} \times M} h(\nabla_h \psi, \nabla_h \psi) dV_h \geq \mu_1 \int_{\{t\} \times M} \psi^2 dV_h.$$

The claim then follows from Proposition 3.3 and (18). \square

4. THE SPECTRAL THEORY OF MANIFOLDS WITH CUSPS

Assumption 4.1. *In this section, we assume $a \leq -1$, $b > 0$, and $c > 0$.*

The manifold $I \setminus \{0\} \times M$ equipped with the metric g_0 of (1) is a disjoint union of ‘cusps’. Here we use the $\epsilon = 0$ case of the analysis in the previous section to derive results about manifolds with cusps.

Let $I^+ \subset \mathbb{R}^+$ be an interval with lower endpoint equal to zero. The manifold $I^+ \times M$ equipped with the metric

$$(21) \quad g = (ct)^{-2a} dt^2 + (ct)^{2b} h.$$

will be called a *cusp* of type (a, b, c) . Note that the limiting manifold (N^0, g_0) of an (a, b) degenerating family is a manifold with ends isometric to cusps.

In the following, we let $\sigma_{ess}(T)$ denote the essential spectrum of an operator T .

Proposition 4.2. *Let (N, g) be a d -dimensional Riemannian manifold with finitely many ends each of which is a cusp of type (a_j, b_j, c_j) . If for some j , we have $a_j < -1$, then $\sigma_{ess}(\Delta_g) = [0, \infty[$. Otherwise each $a_j = -1$, and we have $\sigma_{ess}(\Delta_g) = [m, \infty[$ where m is the infimum of*

$$(22) \quad \left(\frac{c_j \cdot b_j \cdot d_j}{2} \right)^2.$$

Proof. By assumption $a \leq -1$, and hence (N, g) is Riemannian complete [Mlr] §8.1. Therefore $\sigma_{ess}(\Delta)$ depends only on the geometry of the ends. (See, for example, [DnlLi79] Proposition 2.1). In particular, let Δ_j be the Friedrichs extension of Δ restricted to smooth functions supported in the j^{th} end with respect to the L^2 -norm induced by dV_g . Then

$$(23) \quad \sigma_{ess}(\Delta) = \cup_j \sigma_{ess}(\Delta_j).$$

Hence the claim reduces to the consideration of a single cusp. From (10) we have that

$$(24) \quad \sigma_{ess}(\Delta) = \sigma_{ess}(-L) \cup \sigma_{ess}(\Delta^\perp).$$

Hence the claim is a consequence of Propositions 4.4 and 4.3 below. \square

Proposition 4.3. *If $a = -1$, then $\sigma_{ess}(-L) = [(2^{-1} \cdot c \cdot b \cdot d)^2, \infty[$. If $a < -1$, then $\sigma_{ess}(-L) = [0, \infty[$.*

Proof. Let U be as in Proposition 3.2 where $\epsilon = 0$. Since U is unitary, we have $\sigma_{ess}(-L) = \sigma(-U^{-1} \circ L \circ U)$. Moreover, since $\rho(0, t) = ct$, $\rho''_{tt} = 0$ and $(\rho'_t)^2 = c^2$. Thus, with $a = -1$ and $\epsilon = 0$, identity (14) specializes to

$$(25) \quad U^{-1} \circ L \circ U = \partial_s^2 - \left(\frac{c \cdot b \cdot d}{2} \right)^2.$$

The first claim follows.

If $a < -1$, then (14) becomes

$$(26) \quad U^{-1} \circ L \circ U = \partial_s^2 - k \cdot (\alpha^{-1})^* ((ct)^{-2a-2})(s).$$

where $k = 4^{-1} \cdot bd \cdot (bd - 2a - 2)$. From (12) we have $\alpha(t) \sim (a+1)^{-1} \cdot c^a \cdot t^{a+1}$ as $t \rightarrow 0$. Thus,

$$(27) \quad (\alpha^{-1})^*((ct)^{-2a-2})(s) \sim (c \cdot (a+1) \cdot s)^{-2}$$

as s tends to $-\infty$. Since s^{-2} belongs to $L^2([-\infty, -1], ds)$, the second claim follows from standard results on the essential spectrum of Schrödinger operators. See, for example, Theorem XIII.15 [RdSmn]. \square

The proof of the following is modeled on the proof of Lemma 8.7 in [LaxPhl].

Proposition 4.4. *The operator Δ^\perp is compactly resolved. Hence $\sigma_{ess}(\Delta^\perp) = \emptyset$.*

Proof. Without loss of generality, the homogeneity constant c equals 1. Note that the operator Δ^\perp is a Friedrichs extension associated to the form

$$(28) \quad Q(f) = \int_{I^+ \times M} (t^{-2a} \cdot (\partial_t f)^2 + t^{-2b} \cdot h(\nabla_h f, \nabla_h f)) \, dV_g.$$

In particular, to prove the claim it will suffice to show that the intersection of the Q -unit ball with $L^2_\perp(I^+ \times M, dV_g)$ is compact in $L^2_\perp(I^+ \times M, dV_g)$. (See, for example, Theorem XIII.64 [RdSmn]).

We claim that for any $\delta \in I^+ = [0, t_0]$ and any (smooth) function ϕ with $Q(\phi, \phi) \leq 1$ and $\phi_0 \equiv 0$, we have

$$(29) \quad \int_{[0, \delta] \times M} \phi^2 \, dV_g \leq \frac{\delta^{2b}}{\mu_1}$$

where μ_1 is the smallest nonzero eigenvalue of Δ_h . Indeed, (20) holds and hence

$$(30) \quad \mu_1 \cdot \int_{[0, \delta] \times M} \phi^2 \, dV_g \leq \int_{[0, \delta] \times M} h(\nabla_h \phi, \nabla_h \phi) \, dV_g$$

$$(31) \quad = \delta^{2b} \int_{[0, \delta] \times M} t^{-2b} \cdot h(\nabla_h \phi, \nabla_h \phi) \, dV_g.$$

The claim then follows from inspecting (28).

Now let f_i be a sequence of functions such that $Q(f_i, f_i) \leq 1$. Since $b > 0$, given $\epsilon > 0$ small, there exists $\delta < t_0$ such that $\delta^{2b}/\mu_1 = \epsilon/2$. By Rellich's theorem, the restriction of f_i restricted to $[\delta, t_0] \times M$ is an L^2 Cauchy sequence. In particular, there exists N such that if $i, j > N$, then

$$(32) \quad \int_{[\delta, t_0] \times M} (f_i - f_j)^2 \, dV_g \leq \frac{\epsilon}{2}.$$

Applying (29) to $\phi = f_i - f_j$ then gives the claim. \square

The following Proposition gives the decay of ‘cusp forms’.

Proposition 4.5. *Let (N, g) be a Riemannian manifold with a cusp $(I \times M, g)$. Let $\psi \in L^2(N, dV_g)$ be a Laplace eigenfunction of Δ_g with $\psi_0 \equiv 0$ on $I \times M$. Then $\|\psi\|_M^2$ is convex and for all $j > 0$*

$$(33) \quad \lim_{t \rightarrow 0} t^{-j} \cdot \|\psi\|_M^2(t) = 0.$$

Proof. Since $b > 0$, by Lemma 3.4 we have that for any $k > 0$, there exists a $t_* > 0$ such that for all $t \in [0, t_*]$

$$(34) \quad L(\|\psi\|_M^2) \geq k^2 \|\psi\|_M^2.$$

For U defined as in (13), let $f = U^{-1}(\|\psi\|_M^2)$. Note that we have

$$(35) \quad h \geq g \text{ if and only if } U(h) \geq U(g).$$

It follows that (34) holds iff

$$(36) \quad U^{-1} \circ L \circ U(f) \geq k^2 f.$$

Since $\psi \in L^2(N, dV_g)$ and U is a unitary transformation, we have $f \in L^2(-\infty, \alpha(t_0)], ds)$. Moreover, since $b > 0$ and $a \leq -1$, from (14) we have

$$(37) \quad U^{-1} \circ L \circ U f = \partial_s^2 f - \phi \cdot f$$

where $\phi \geq 0$. Thus, it follows from (36) that f is convex and that $f(s) = O(e^{ks})$ as $s \rightarrow -\infty$.

Note that if h is a linear, then since $b > 0$, and $a \leq -1$, the function $U(h)$ is also convex. Thus, since f is convex, it follows from (35) that $U(f) = \|\psi\|_M^2$ is convex. Moreover, from the definition of U , we find that $U(f) = \|\psi\|_M^2$ satisfies (33) with $j = k - \frac{bd}{2}$. \square

Lemma 4.6. *Let (N, g) be a manifold with a cusp $(I \times M, g)$. Let $\psi \in L^2(N, dV_g)$ be a Laplace eigenfunction of Δ_g with $\psi_0 \equiv 0$ on $I \times M$. Then $\partial_t \psi$ belongs to $L^2(I \times M, dV_g)$.*

Proof. Since $b > 0$, there exists $t_* > 0$ such that for $0 < t < t_*$, we have $\lambda - \mu_1 t^{-2b} < 0$. It then follows from Proposition 3.3, (20), and (18), that for $0 < t < t_*$

$$(38) \quad (ct)^{-2a} \int_M (\partial_t \psi)^2 \leq \frac{1}{2} \cdot L(\|\psi\|_M^2).$$

Multiply both sides by t^{2a} and integrate over $I_s = [s, t_*]$ to obtain

$$(39) \quad \int_{I_s \times M} (\partial_t \psi)^2 dV_g \leq \frac{c^{2a}}{2} \cdot \int_s^{t_*} t^{2a} \cdot L(\|\psi\|_M^2) t^{a+bd} dt.$$

Thus, it suffices to show that the integral on the right hand side remains bounded as t tends to zero. To verify this, we integrate by parts and obtain

$$(40) \quad \int_s^{t_*} t^k \cdot L(\|\psi\|_M^2) = (t^\alpha \cdot \partial_t \|\psi\|_M^2 + C_1 \cdot t^\beta \|\psi\|_M^2) \Big|_s^{t_*} + C_2 \cdot \int_s^{t_*} t^\gamma \cdot \|\psi\|_M^2.$$

where C_1 , C_2 , α , β , and γ are constants that depend on k . By Proposition 4.5, $\|\psi\|_M^2$ is convex and satisfies (33). It follows that both $t^\beta \|\psi\|_M^2$ and $t^\gamma \partial_t \|\psi\|_M^2$ are bounded for sufficiently small t . The claim follows. \square

5. CONVERGENCE OF EIGENFUNCTIONS

Theorem 5.1. *Let (N, g_ϵ) be a (a, b) -degenerating family with $a \leq -1$ and $b > 0$. For $\epsilon_j \rightarrow 0$, let ψ_j be a sequence of eigenfunctions of Δ_{ϵ_j} on $L^2(N, dV_g)$ with eigenvalues $\lambda_j \leq \Lambda$. Then there is a subsequence ψ_k , a sequence $a_k \in \mathbb{R}$, and a nontrivial eigenfunction ψ_* of the C^∞ Laplacian on N^0 , such that for every compact $A \subset N^0$, the sequence $a_k \psi_k$ converges to ψ_* in $H^1(A, dV_{g_0})$. Moreover, if $(\psi_k)_0 \equiv 0$ for each m , then $a_k \psi_k$ converges to ψ_* in $L^2_\perp(N^0, dV_{g_0})$.*

Proof. For $f \in C^0(N)$, define

$$(41) \quad \bar{f}(x) = \begin{cases} f(x) & x \in N \setminus (\frac{1}{3}I \times M) \\ f(x) - f_0(x) & x \in \frac{1}{3}I \times M \end{cases}$$

where f_0 is the constant mode of f . Now define

$$(42) \quad \phi_j = \frac{\psi_j}{\|\psi_j\|}$$

where $\|\cdot\|$ is the $L^2(N, dV_g)$ norm. We will show that this renormalized sequence satisfies the claim.

Let $\chi \in C_0^\infty(N)$ be nonnegative, have support in $N \setminus (\frac{1}{3}I \times M)$, and satisfy $\chi \equiv 1$ on $N \setminus (\frac{2}{3}I \times M)$. Using the Schwarz inequality (for both TM and L^2) and the fact that $Q(\phi_j) = \lambda_j \cdot \|\phi_j\|$, we find that

$$(43) \quad Q(\chi \cdot \phi_j) \leq \left(\left(\int_N \chi^2 |\nabla \phi_j|^2 dV_g \right)^{\frac{1}{2}} + \left(\int_N \phi_j^2 |\nabla \chi|^2 dV_g \right)^{\frac{1}{2}} \right)^2$$

Using integration by parts and the fact that $\Delta \phi_j = \lambda_j \cdot \phi_j$, one finds a constant C depending only on χ such that $\int_N \chi^2 |\nabla \phi_j|^2$ is less than $C \cdot (\Lambda + C)$ times the integral of ϕ_j^2 over $N \setminus (\frac{1}{3}I \times M)$. By (41), this latter integral is less than 1, and hence $Q(\chi \cdot \phi_j)$ is a bounded sequence in the space of $H^1(N, dV_g)$ functions that have support outside of $\frac{1}{3}I \times M$. By Rellich's theorem we may pass to a subsequence such that $\chi \cdot \phi_j$ converges in this space to a function $\chi \cdot \phi_*$. Hence the restriction of ϕ_j to $N \setminus (\frac{2}{3}I \times M)$ converges in H^1 .

It follows that the restriction of the constant mode, $(\phi_j)_0$, to $I \setminus (\frac{2}{3}I)$ converges to $(\phi_*)_0$ in H^1 norm. Thus, by Sobolev's embedding theorem (in dimension 1) $(\phi_j)_0$ converges to $(\phi_*)_0$ in C^0 , and hence the boundary conditions of (11) on $I \setminus K$ converge in C^0 as $j \rightarrow \infty$ for each nontrivial interval $0 \in K \subset I$. Since the metrics converge, the coefficients of (11) converge in C^0 . It follows that $(\phi_*)_0$ extends uniquely to $I \setminus \{0\}$, and that, for each $K \ni 0$, the sequence $(\phi_j)_0$ converges to $(\phi_*)_0$ in $C^0(K \setminus I)$.

Since ϕ_j is obtained from $\bar{\phi}_j$ by adding on the constant mode $(\phi_j)_0$, we find that for every interval $K \subset I$, the restriction of ϕ_j to $N \setminus (K \times M)$ is uniformly bounded in L^2 -norm. Hence, for each $K \subset I$, the argument above can be applied to give a further subsequence ϕ_j whose restriction converges in $H^1(N \setminus (K \times M), dV_g)$. Diagonalization yields a further subsequence that converges to a function ϕ_* for every K .

Since λ_j is bounded, we may take a further subsequence such that λ_j converges to some $\lambda_* \geq 0$. For each test function T supported away from $\{0\} \times M$, we have that $Q_{\epsilon_j}(\phi_j, T) \rightarrow Q_0(\phi_*, T)$. It follows that ϕ_* is a weak—and hence by elliptic regularity, a strong—solution to $\Delta_0 \phi_* = \lambda_* \cdot \phi_*$.

It remains to show that ϕ_* does not vanish identically. Since $b > 0$ and $a \leq -1$, there exists an interval $J \subset \frac{1}{3}I$ symmetric about 0 such that for all $t \in J$ and ϵ sufficiently small

$$(44) \quad \rho(\epsilon, t)^{2b} \leq \frac{2\mu_1}{\Lambda}$$

and

$$(45) \quad L(\rho^{2b})(\epsilon, t) \leq \frac{\mu_1}{2}.$$

Indeed, the operator L adds $-2a - 2 \geq 0$ to the degree of homogeneity. Let $\chi \in C_0^\infty(J)$ with $\chi \equiv 1$ on $\frac{1}{2}J$. From (44), Lemma 3.4, and the self-adjointness of L , we obtain:

$$(46) \quad \begin{aligned} \mu_1 \cdot \int_{J \times M} \chi \cdot \bar{\phi}_j^2 dV_g &= 2 \int_J \chi \cdot \rho^{2b} \cdot (\mu_1 \rho^{-2b} - \lambda_j) \|\bar{\phi}_j\|_M^2 \rho^{a+bd} dt \\ &\leq \int_J \chi \cdot \rho^{2b} \cdot L(\|\bar{\phi}_j\|_M^2) \rho^{a+bd} dt \\ &= \int_{J \times M} L(\chi \cdot \rho^{2b}) \cdot \bar{\phi}_j^2 dV_g. \end{aligned}$$

Note that $L(\chi \cdot \rho^{2b})$ equals $\chi \cdot L(\rho)$ plus a smooth function supported on $J \setminus \frac{1}{2}J$ bounded by C . Thus, it follows from (45) and (46) that

$$(47) \quad \frac{\mu_1}{2} \int_{\frac{1}{2}J \times M} \bar{\phi}_j^2 dV_g \leq C \cdot \int_{(J \setminus \frac{1}{2}J) \times M} \bar{\phi}_j^2 dV_g.$$

Thus, from the normalization of the L^2 -norm in (42), we find that

$$(48) \quad 1 \leq \frac{2C}{\mu_1} \int_{(J \setminus \frac{1}{2}J) \times M} \bar{\phi}_j^2 dV_g + \int_{N \setminus (\frac{1}{2}J \times M)} \bar{\phi}_j^2 dV_g.$$

Therefore, since ϕ_j restricted to $N \setminus (\frac{1}{2}J \times M)$ converges in L^2 to ϕ_* , the function ϕ_* is nontrivial. \square

6. EIGENVALUE CONTINUITY FOR Δ^\perp

In this section, g_ϵ is a family of (a, b) -degenerating metrics on $I \times M$ that depends continuously on ϵ . By Proposition 4.4, for each ϵ —including $\epsilon = 0$ —the operator Δ_ϵ^\perp has a countable collection of eigenvalues (including multiplicities)

$$(49) \quad 0 < \lambda_1(\epsilon) \leq \lambda_2(\epsilon) \leq \lambda_3(\epsilon) \leq \dots$$

Theorem 6.1. *Let $a \leq -1$ and $b > 0$. Then for each i , the function $\epsilon \rightarrow \lambda_i(\epsilon)$ is continuous.*

Proof. The operator Δ_ϵ^\perp is defined via the (sesquilinear) form

$$(50) \quad Q_\epsilon(f) = \int \rho^{-2a} \cdot (\partial_t f)^2 \cdot \rho^{a+bd} dV_h + \int \rho^{-2b} \cdot h(\nabla_h f, \nabla_h f) \cdot \rho^{a+bd} dV_h$$

restricted to $C_0^\infty(I \times M) \cap L_\perp^2(I \times M, dV_{g(\epsilon)})$. For $\epsilon \neq 0$, the domain of Δ_ϵ^\perp is $H^1(I \times M, dV_g) \cap L_\perp^2(I \times M, dV_{g(\epsilon)})$. For any power c , the function ρ^c is uniformly continuous in ϵ over closed intervals that do not contain 0. It follows from §VI.3.2 [Kat] that Δ_ϵ^\perp is continuous in the ‘generalized sense’ for $\epsilon \neq 0$. Thus, by §IV.3.5

[Kat], any finite system of eigenvalues varies continuously for $\epsilon \neq 0$. It follows that each λ_i is continuous for $\epsilon \neq 0$.

We are left with showing the continuity at $\epsilon = 0$. Let $V_{k-1}(\epsilon)$ be a span of eigenfunctions associated to the first k eigenvalues: $\lambda_1(\epsilon), \dots, \lambda_k(\epsilon)$. Then by the minimax principle, $\lambda_{k+1}(\epsilon)$ is the minimum value of the functional $F_\epsilon(\phi) = Q_\epsilon(\phi)/\|\phi\|_\epsilon^2$ over $V_{k-1}(\epsilon)^\perp$ where $\|\cdot\|_\epsilon$ denotes the $L^2(I \times M, dV_{g_\epsilon})$ -norm.

Let ϕ_1 be an eigenfunction of Δ_0^\perp with eigenvalue $\lambda_1(0)$. By Proposition 4.6, the function ϕ_1 belongs to the domain of F_ϵ for small ϵ . From the continuity of ρ , we find that

$$\lim_{\epsilon \rightarrow 0} F_\epsilon(\phi_1) = F_0(\phi_1) = \lambda_1(0).$$

Hence, by the minimax principle, $\limsup_{\epsilon \rightarrow 0} \lambda_1(\epsilon) \leq \lambda_1(0)$. For each ϵ , let $\phi_1(\epsilon)$ be an eigenfunction of Δ_ϵ^\perp with eigenvalue $\lambda_1(\epsilon)$. By applying Theorem 5.1 to a subsequence whose eigenvalues limit to $\liminf_{\epsilon \rightarrow 0} \lambda_1(\epsilon)$, we obtain an eigenfunction ϕ of Δ_0^\perp with eigenvalue $\lambda_* = \liminf_{\epsilon \rightarrow 0} \lambda_1(\epsilon)$ less than or equal to λ_1 . But since λ_1 is the smallest eigenvalue, $\lambda_* = \lambda_1$. It follows that $\lim_{\epsilon \rightarrow 0} \lambda_1(\epsilon) = \lambda_1(0)$.

The continuity at $\epsilon = 0$ of λ_k for general k follows from a straightforward inductive argument involving $V_k(\epsilon)$. (Note that no claim is made about the continuity of the family $V_k(\epsilon)$). \square

Remark 6.2. With minor modifications, the proof of Theorem 6.1 gives the continuity of the eigenvalues of the *cut-off (or pseudo-)Laplacian* defined as in [Ji93].

7. COUNTING RELATIVELY SMALL EIGENVALUES

In the following $O(1)$ will denote a bounded function of ϵ .

Theorem 7.1. *Let $a = -1$ and $b > 0$. For $\Lambda > 0$, let $\mathcal{N}_\Lambda(\epsilon)$ be the number of eigenvalues of Δ_{g_ϵ} that lie in $[0, \Lambda]$. Then*

$$(51) \quad \mathcal{N}_\Lambda(\epsilon) = \left(c_+ \sqrt{\Lambda - \left(\frac{c_+ \cdot b \cdot d}{2} \right)^2} + c_- \sqrt{\Lambda - \left(\frac{c_- \cdot b \cdot d}{2} \right)^2} \right) \cdot \frac{\log(\epsilon^{-1})}{\pi} + O_\Lambda(1).$$

Here the value of the square root is taken to be zero if the argument is negative.

Proof. By the Dirichlet monotonicity, we have that $\mathcal{N}_\Lambda(\epsilon)$ is bounded below by the number, $\mathcal{N}_\Lambda^D(\epsilon)$, of eigenvalues of the Dirichlet problem on $I \times M$. By Neumann monotonicity, $\mathcal{N}_\Lambda(\epsilon)$ is bounded above by the number, $\mathcal{N}_\Lambda^N(\epsilon)$, of eigenvalues of the Neumann problem on $I \times M$ plus the number of eigenvalues of the Neumann problem on $N \setminus (I \times M)$. The latter is $O_\Lambda(1)$ since $g(\epsilon)$ converges uniformly on $N \setminus (I \times M)$. In sum, we have

$$(52) \quad \mathcal{N}_\Lambda^D(\epsilon) \leq \mathcal{N}_\Lambda(\epsilon) \leq \mathcal{N}_\Lambda^N(\epsilon) + O_\Lambda(1).$$

By (10) the Dirichlet (resp. Neumann) spectrum decomposes into the Dirichlet (resp. Neumann) eigenvalues of L acting on $L^2(I, \rho^{n-2} dt)$ and those of Δ_ϵ^\perp acting on $L^2_\perp(I \times M, dV_{g_\epsilon})$. By Theorem 6.1, the number of Dirichlet (resp. Neumann) eigenvalues of Δ_ϵ^\perp is $O_\Lambda(1)$. Hence, the claim reduces to the Lemma 7.2 below. \square

To emphasize its dependence on ϵ , we let L_ϵ denote the operator defined in (5).

Lemma 7.2. *For each $\epsilon > 0$, let $\mathcal{N}_\Lambda^\partial(\epsilon)$ be the number of solutions $\lambda \in [0, \Lambda]$ to the Dirichlet (resp. Neumann) eigenvalue problem*

$$(53) \quad -L_\epsilon v = \lambda v$$

on the interval I . Then $\mathcal{N}_\Lambda^\partial(\epsilon)$ satisfies (51).

Proof. First note that since $a = -1$ and ρ is homogeneous of degree 1, the eigenvalue problem (53) on I is equivalent to the problem

$$(54) \quad L_1 v = -\lambda v$$

on the dilated interval $\epsilon^{-1}I$. By conjugating both sides of (54) by $U = U_1$ and applying Proposition 3.2, we obtain the equivalent eigenvalue problem

$$(55) \quad \partial_t^2 u - \frac{bd}{2}(\alpha_1^{-1})^* \left(\rho \cdot \rho''_{tt} + \frac{bd}{2}(\rho'_t)^2 \right) u = -\lambda u.$$

on the interval $\alpha_1(\epsilon^{-1}I)$.

Let $a_-(\epsilon) < a_+(\epsilon)$ be the endpoints of the interval $\alpha_1(\epsilon^{-1}I)$. Let $\mathcal{N}_\Lambda^+(\epsilon)$ (resp. $\mathcal{N}_\Lambda^-(\epsilon)$) be the number of eigenvalues λ of (55) on the interval $[0, a_+]$ (resp. $[a_-, 0]$). Then by Dirichlet-Neumann bracketing (see, for example, [CrnHlb] p 408-409), we have

$$(56) \quad \mathcal{N}_\Lambda^\partial(\epsilon) = \mathcal{N}_\Lambda^+(\epsilon) + \mathcal{N}_\Lambda^-(\epsilon) + O_\Lambda(1).$$

We claim that it suffices to show that

$$(57) \quad a_\pm(\epsilon) \sim \pm c_\pm \ln(\epsilon^{-1}),$$

as ϵ tends to zero, and that

$$(58) \quad \frac{bd}{2}(\alpha_1^{-1})^* \left(\rho \cdot \rho''_{tt} + \frac{bd}{2}(\rho'_t)^2 \right) = \left(\frac{c_\pm \cdot b \cdot d}{2} \right)^2 \cdot u + r(s)$$

where $r(s)$ is $O(|s|^{-2})$ for $|s|$ large. Indeed, then (55) would be equivalent to

$$(59) \quad \partial_s^2 u = \left(-\lambda + \left(\frac{c_\pm \cdot b \cdot d}{2} \right)^2 \right) u + r(s).$$

By Proposition 7.3 below, we would then have

$$(60) \quad \mathcal{N}_\Lambda^\pm(\epsilon) = a_\pm(\epsilon) \sqrt{\Lambda - \left(\frac{c_\pm \cdot b \cdot d}{2} \right)^2},$$

and the claim would follow from (56).

To verify (57) and (58), we use homogeneity. We have $\rho^{-1}(1, t) \sim c_\pm t^{-1}$, and hence $\alpha_1(t) \sim c_\pm \ln(t)$ as $t \rightarrow \pm\infty$. Estimate (57) follows. Moreover,

$$(61) \quad \alpha_1^{-1}(t) \sim \exp(c_\pm t).$$

To prove (58), we use the following fact: If $\sigma(\epsilon, t)$ is smooth, homogeneous of degree k , and $\sigma(0, t) \equiv 0$, then $\sigma(1, t) = O(|t|^{-k+1})$ as $|t|$ tends to infinity. (Indeed, we have $h(1, t) = t^{-k}h(t^{-1}, 1)$ and Taylor expand in the first coordinate.) By applying this fact to ρ''_{tt} and $(\rho'_t)^2 - c_\pm^2$, we find that

$$(62) \quad \rho \cdot \rho''_{tt}(1, t) = O(|t|^{-1})$$

and

$$(63) \quad (\rho'_t)^2 = c_\pm^2 + O(|t|^{-1}).$$

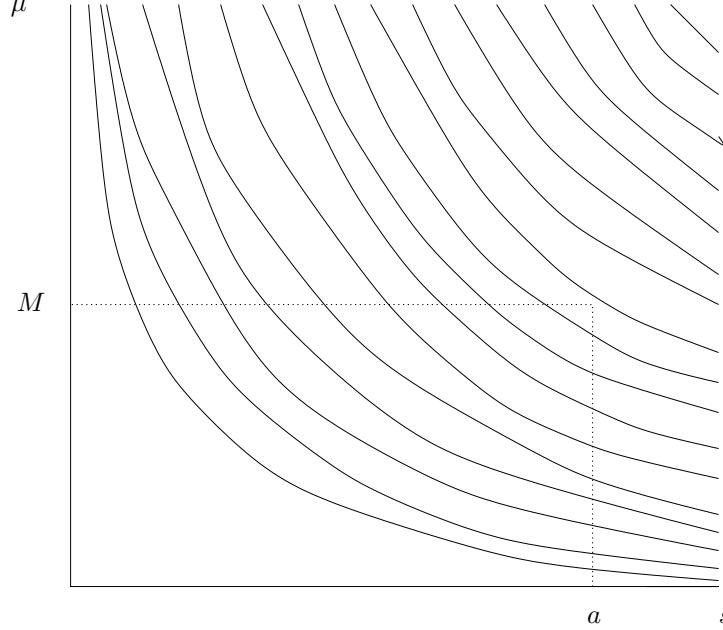


FIGURE 2. Sturm-Liouville eigenvalues on $[0, a]$ as functions of a .

Equation (58) then follows from (61). \square

The following appears in [ChvDdz94]. To make our exposition self-contained, we include it here with an alternate proof.

Proposition 7.3. *Let $r \in C^0(\mathbb{R})$ be integrable with respect to Lebesgue measure. Let $\mathcal{N}(a, b)$ be the number of solutions $\mu \in [0, M]$ to the Dirichlet (resp. Neumann) eigenvalue problem*

$$(64) \quad \partial_s^2 u = -\mu u + ru$$

on $[0, a]$. Then

$$(65) \quad \mathcal{N}_M(a) = \frac{a\sqrt{M}}{\pi} + O_M(1)$$

where $O_M(1)$ is a bounded function of $a \in \mathbb{R}$.

Proof. Let $0 = \mu_0(a) < \mu_1(a) < \mu_2(a) < \dots$ denote the (necessarily simple) Dirichlet (resp. Neumann) eigenvalues of (64) for the interval $[0, a]$. By the standard theory, for each k , $\mu_k(a)$ is a decreasing function of $a > 0$ with $\lim_{a \rightarrow 0} \mu_k(a) = +\infty$. It follows that

$$(66) \quad \mathcal{N}_M(a) = \text{Card}\{k \in \mathbb{N} : \mu_k(b) = M \text{ and } b \leq a\}.$$

See Figure 2.

Let $v \in C^2(\mathbb{R})$ be a nonzero solution to (64) with $\mu = M$ and $v(0) = 0$ (resp. $v'(0) = 0$). (The function v need not satisfy the Dirichlet (Neumann) boundary

condition at a). Since r is integrable, there exist (see, for example, [CrnHlb] p. 332-333) bounded functions $\alpha, \delta \in C^1(\mathbb{R})$, $\delta'(s) > 0$, such that

$$(67) \quad v(s) = \alpha(s) \cdot \sin(\sqrt{M}s + \delta(s)).$$

Note that $\mu_k(b) = M$ if and only if $v(b) = 0$ (resp. $v'(b) = 0$). Thus by (67), a Dirichlet eigenvalue $\mu_k(b) = M$ if and only if $\pi^{-1}(\sqrt{M}b + \delta(b)) \in \mathbb{Z}$ (resp. $\pi^{-1}(\sqrt{M}b + \delta(b)) \in \mathbb{Z} + \frac{\pi}{2}$).

In sum, for the Dirichlet problem,

$$(68) \quad N_M(a) = \text{Card}\{0 \leq b \leq a : \pi^{-1}(\sqrt{M}b + \delta(b)) \in \mathbb{Z}\}.$$

Thus, letting $C = \sup |\delta(s)|$ we have

$$(69) \quad \pi^{-1}(\sqrt{M}b - C) - 1 \leq N_M(a) \leq \pi^{-1}(\sqrt{M}b + C),$$

and the claim follows for the Dirichlet case. To prove the Neumann case, one can use a similar analysis involving differentiating (67). Or one can derive the Neumann case from the Dirichlet case via Neumann-Dirichlet bracketing for Sturm-Liouville problems: the $k+2$ nd Neumann eigenvalue is at least as great as the k th Dirichlet eigenvalue [Wnb]. \square

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INDIANA UNIVERSITY, BLOOMINGTON, IN.

E-mail address: cjudge@indiana.edu, <http://www.math.indiana.edu>